Recall: $N : S \longrightarrow S^2$ Gauss map

$$S_p := -dN_p : T_pS \longrightarrow T_{N(p)}S^2 \cong T_pS$$ Shape operator

this is just a convention.

Def$^{\underline{n}}$: The second fundamental form (at p) with respect to N is the symmetric bilinear form

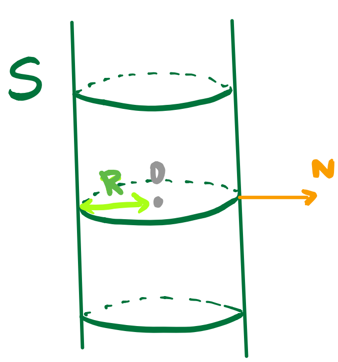
$$A : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{A(u,v) := \langle Su, v \rangle}$$

Note: In local coord., $T_pS = \text{span}\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$

$$A = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{\text{symmetric matrix}} \quad \text{where} \quad A_{ij} := A\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

(3) Cylinder $\quad S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\}$



$$N(x,y,z) = \frac{1}{R}(x,y,0)$$

Question: How to compute

$$S = -dN \ ?$$

Ans: Use local coordinates!

Locally, we can parametrize the cylinder by "cylindrical coordinates"

$$\mathbf{X}(\theta, z) := (R\cos\theta,\ R\sin\theta,\ z)$$

At each point on S, T_pS is spanned by

$$\begin{cases} \dfrac{\partial}{\partial\theta} := \dfrac{\partial \mathbf{X}}{\partial\theta} = (-R\sin\theta,\ R\cos\theta,\ 0) \\ \dfrac{\partial}{\partial z} := \dfrac{\partial \mathbf{X}}{\partial z} = (0,\ 0,\ 1) \end{cases}$$

Hence, a (local) unit normal vector field is

$$N(\theta, z) = \frac{\frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z}}{\left\| \frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z} \right\|} = (\cos\theta,\ \sin\theta,\ 0)$$

$$\Rightarrow \begin{cases} dN\left(\dfrac{\partial}{\partial\theta}\right) = \dfrac{\partial N}{\partial\theta} = (-\sin\theta,\ \cos\theta,\ 0) \\ dN\left(\dfrac{\partial}{\partial z}\right) = \dfrac{\partial N}{\partial z} = (0,\ 0,\ 0) \end{cases}$$

In terms of the basis $\left\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right\}$ for T_pS

$$S = -dN = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{H \equiv -\frac{1}{R}, \qquad K \equiv 0}$$

<u>Note</u>: In all the examples above, S are represented by symmetric matrices. This is in fact a general phenomena.

<u>Prop</u>: $S = -dN_p : T_pS \to T_pS$ is a self adjoint operator on the inner product space $(T_pS, \langle , \rangle)$.

<u>Proof</u>: Take any parametrization $\mathbf{X}(u,v)$ near p

$$T_pS = \operatorname{span}\left\{ \frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v} \right\}$$

It suffices to prove

$$\boxed{\left\langle S\left(\frac{\partial \mathbf{X}}{\partial u}\right), \frac{\partial \mathbf{X}}{\partial v} \right\rangle = \left\langle \frac{\partial \mathbf{X}}{\partial u}, S\left(\frac{\partial \mathbf{X}}{\partial v}\right) \right\rangle} \quad (*)$$

By abuse of notation, we write

$$N(u,v) := N \circ \mathbf{X}(u,v)$$

Since $N_p \perp T_pS$ <u>for all $p \in S$</u>,

$$\left\langle N, \frac{\partial \mathbf{X}}{\partial v} \right\rangle \equiv 0$$

$$\xRightarrow[\text{w.r.t. } u]{\text{differentiate}} \quad \left\langle \underbrace{\frac{\partial N}{\partial u}}_{dN\left(\frac{\partial \mathbf{X}}{\partial u}\right) = -S\left(\frac{\partial \mathbf{X}}{\partial u}\right)}, \frac{\partial \mathbf{X}}{\partial v} \right\rangle + \left\langle N, \frac{\partial^2 \mathbf{X}}{\partial u \partial v} \right\rangle \equiv 0$$

Hence, $\quad \left\langle S\left(\frac{\partial \mathbf{X}}{\partial u}\right), \frac{\partial \mathbf{X}}{\partial v} \right\rangle = \left\langle N, \frac{\partial^2 \mathbf{X}}{\partial u \partial v} \right\rangle$

Similarly, $\quad \left\langle N, \frac{\partial \mathbf{X}}{\partial u} \right\rangle \equiv 0$

$$\xRightarrow[\text{w.r.t. } v]{\text{differentiate}} \left\langle S\left(\frac{\partial \mathbf{X}}{\partial v}\right), \frac{\partial \mathbf{X}}{\partial u} \right\rangle = \left\langle N, \frac{\partial^2 \mathbf{X}}{\partial v \partial u} \right\rangle$$

$$\boxed{\frac{\partial^2 \mathbf{X}}{\partial u \partial v} = \frac{\partial^2 \mathbf{X}}{\partial v \partial u}} \Longrightarrow (*)$$

"mixed partials are the same"

$\square$

§ Local expressions of curvatures

Suppose we have a parametrization $X(u_1, u_2)$ of S.

We can express the 1^{st} & 2^{nd} f.f. of S as 2×2 symmetric matrices (under the basis $\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$ of T_pS)

$$(g_{ij}) = \begin{pmatrix} \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \rangle \\ \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2} \rangle \end{pmatrix}$$

$$(A_{ij}) = \begin{pmatrix} \langle \frac{\partial X}{\partial u_1}, -\frac{\partial N}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_1}, -\frac{\partial N}{\partial u_2} \rangle \\ \langle \frac{\partial X}{\partial u_2}, -\frac{\partial N}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_2}, -\frac{\partial N}{\partial u_2} \rangle \end{pmatrix}$$

Note: "Differentiate by part", we can also write

$$\boxed{\langle \frac{\partial X}{\partial u_i}, -\frac{\partial N}{\partial u_j} \rangle = \langle \frac{\partial^2 X}{\partial u_i \partial u_j}, N \rangle}$$

This is usually easier to calculate!

Question: Can we compute the curvatures K and H in terms of (g_{ij}) and (A_{ij})?

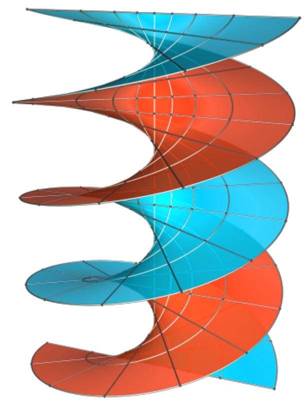
Yes!

Formula : $\boxed{K = \dfrac{\det(A_{ij})}{\det(g_{ij})}}$ & $\boxed{H = tr((g_{ij})^{-1}(A_{ij}))}$

These formulas are very useful for computations!

Example: (Helicoid)

$$\mathbf{X}(u,v) = (v\cos u, v\sin u, u)$$



$$\frac{\partial \mathbf{X}}{\partial u} = (-v\sin u, v\cos u, 1)$$

$$\frac{\partial \mathbf{X}}{\partial v} = (\cos u, \sin u, 0)$$

1st f.f. $$(g_{ij}) = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N = \frac{\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}}{\left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\|} = \frac{(-\sin u, \cos u, -v)}{\sqrt{1+v^2}}$$

$$\frac{\partial^2 \mathbf{X}}{\partial u^2} = (-v\cos u, -v\sin u, 0)$$

(Note: Compute $\frac{\partial N}{\partial v}$ is very complicated!)

$$\frac{\partial^2 \mathbf{X}}{\partial u \partial v} = (-\sin u, \cos u, 0)$$

$$\frac{\partial^2 \mathbf{X}}{\partial v^2} = (0,0,0)$$

2nd f.f. $(A_{ij}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+v^2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}$

Hence, the shape operator is given by

$$S = (g_{ij})^{-1}(A_{ij}) = \begin{pmatrix} 0 & \frac{1}{(1+v^2)^{3/2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}$$

$\boxed{K = \det S = -\frac{1}{(1+v^2)^2}}$ & $\boxed{H = \operatorname{tr} S \equiv 0}$